Public Good Provision Games on Networks with Resource Pooling

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Abstract We consider the interaction of strategic agents and their decision-making process toward the provision of a public good. In this interaction, each user exerts a certain level of effort to improve his own utility. At the same time, the agents are interdependent and the utility of each agent depends not only on his own effort but also on the other agents' effort level. As the agents have a limited budget and can exert limited effort, question arises as to whether there is advantage to agents pooling their resources. In this study, we show that resource pooling may or may not improve the agents' utility when they are driven by self-interest. We identify some scenarios where resource pooling does lead to social welfare improvement as compared to without resource pooling. We also propose a taxation–subsidy mechanism that can effectively incentivize the agents to exert socially optimal effort under resource pooling.

Keywords Public good · Resource pooling · Social welfare

1 Introduction

The interactions among strategic agents form a network game [5, 10], and the provision of public goods is one particular type of such games when their decision-making processes concern the provision of a public good [3, 6, 15]. The interdependent security (IDS) game [9, 12, 16] is one such example. Other examples of a network game include networked Cournot competition [4, 7]. The goal in studying these games

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is often to characterize their equilibrium and the effect of the underlying network structure on such equilibrium. Such findings can help policy and network designers make better decisions in inducing desirable outcomes.

The conditions for existence and uniqueness of equilibrium in a network game with linear best-response functions have been studied in [11, 15]. Miura-Ko et al. [11] consider a security decision-making problem and find a sufficient condition for the uniqueness of the equilibrium, while [15] considers the provision of a public good and introduces a necessary and sufficient condition for the uniqueness of the equilibrium. Network games with nonlinear best-response functions have also been studied in the literature [2, 14]. Acemoglu et al. [2] show that if the best-response function is a contraction or non-expansive mapping, then the equilibrium is unique. Naghizadeh et al. [14] provide a condition on the smallest eigenvalue of a matrix composed of interdependency factor and derivative of the best-response function to guarantee the uniqueness of equilibrium. Similarly, a potential game for modeling the Cournot competition is introduced in [1], where it is shown that if the cost function is strictly convex, then the equilibrium is unique, while [4] provides a sufficient condition for uniqueness of equilibrium and studies the effect of the competition structure on the firms' profit.

Of equal interest in the context of network games is the question of designing mechanisms that induce network games with desirable equilibrium properties. For instance, in the literature of IDS games where the public good being provisioned is agents' investment in security, incentive mechanisms have been proposed to induce higher levels of effort by agents. In [9], Grossklags et al. suggest bonus and penalty based on agents' security outcome, while Parameswaran et al. [16] propose a mechanism to overcome the free-riding/underinvestment issue, where an authority monitors user investment.

In this paper, we study a generic problem of public good provision on networks, where each agent/player exerts a certain effort (the provision of a public good) that impacts themselves as well as their neighbors on a connectivity graph. The difference between this and most prior work (including those cited above) is that we assume each agent has a budget constraint but may be allowed to pool their resources. Specifically, we consider three different scenarios:

(i) Agents are not allowed to pool their resources.
(ii) Agents are free to pool their resources but will do so selfishly.
(iii) Agents are free to pool their resources and are incentivized to do so optimally (in terms of social welfare) and voluntarily.

Case (i) is a baseline scenario. The agents' utility in this case is also considered their *outside option* when presented with a mechanism. Case (ii) allows us to investigate the behavior of strategic agents when resource pooling is allowed but not regulated. As we will show the Nash equilibrium in this case may lead to improved as well as worsening social welfare as compared to Case (i). In Case (iii), we design a mechanism to incentivize agents to choose socially optimal actions in resource pooling; this mechanism is budget-balanced, incentive-compatible, and individually rational.

The remainder of the paper is organized as follows. We present the model, preliminaries, and Case (i) in Sect. 2. We then analyze Case (ii) in Sect. 3. We propose a mechanism that attains the socially optimal solution at the equilibrium of its induced game in Sect. 4. We present numerical examples in Sect. 5 and conclude the paper in Sect. 6.

2 Model: A Scenario Without Resource Pooling

We study the interaction of N agents in a directed network $\mathcal{G} = (V, E)$, where $V = \{1, 2, \cdots, N\}$ is a set of N agents and $E = \{(i, j)|i, j \in V\}$ is a set weighted edges between them. An agent $i \in V$ has limited budget B_i and chooses to exert effort $x_i \in [0, B_i]$ toward improving his utility. Agent i's payoff depends on his own effort, as well as the effort exerted by others with nonzero influence on i. An edge $(i, j) \in E$ indicates that agent j depends on agent i (or that agent i influences j) with edge weight $g_{ij} \in [0, 1]$. The dependence need not be symmetrical, i.e., $g_{ij} \neq g_{ji}$ in general. We shall assume $g_{ii} = 1,\ i = 1, 2, \cdots, N$ and $g_{ij} < 1,\ \forall i \neq j$ to reflect the notion that an agent is his own biggest influence.

Let $\boldsymbol{x} = [x_1, x_2, \cdots, x_N]$ be the profile of exerted efforts by all N agents. Then, the utility of agent i is given by,

$$u_i(\boldsymbol{x}) = \sum_{j \in V} g_{ji} \mu_j(x_j)\ , \tag{1}$$

where $\mu_i(a)$ is a function determining the benefit to agent i from effort a.

We will assume that $\mu_i(.)$ is differentiable and strictly increasing for all i and that μ_i is strictly concave. This implies that while the initial effort leads to a considerable increase in utility, the marginal benefit decreases as effort increases.

Since $\mu_i(.)$ is strictly increasing and agents are strategic, the best strategy for each agent is to use all of his budget. Therefore,

$$v_i^o = \sum_{j \in V} g_{ji} \mu_j(B_j) \tag{2}$$

is the highest utility that agent i achieves without resource pooling. We shall take v_i^o to be the participation constraint of agent i in deciding whether there is incentive to participate in any mechanism.

3 Public Good Provision with Resource Pooling

Consider now the scenario where the agents can pool their resources. Let $\boldsymbol{x}_i = [x_{i1}, x_{i2}, \cdots, x_{iN}]^T$ be the action of agent i where $x_{ij} \geq 0$ denotes the effort exerted by agent i on behalf of agent j, e.g., by transferring part of its budget to j.

Let $X = [\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N]^T$ be the profile of the exerted efforts, an $N \times N$ matrix. Due to the increasing nature of μ_j, we have $\sum_{j=1}^{N} x_{ij} = B_i$. As a result, the agent's utility given action profile X is:

$$v_i(X = [\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_N]^T) = \sum_{j \in V} g_{ji} \cdot \mu_j(\sum_{k=1}^{N} x_{kj}) \tag{3}$$

Notice that $v_i(X)$ is concave in X, but it is not necessary strictly concave. Moreover, $v_i(X)$ is strictly concave in $\boldsymbol{x}_i$.

This game is denoted as G_P and given by the tuple $(V, \{x_{ij}|i, j \in V\}, \{v_i(.)|i \in V\})$. Let $B_i(\boldsymbol{x}_{-i})$ denote the best-response function of agent i:

$$\begin{aligned} B_i(\boldsymbol{x}_{-i}) &= \arg\max_{\boldsymbol{x}_i} \sum_{j \in V} g_{ji} \cdot \mu_j(\sum_{k=1}^{N} x_{kj}) \\ \text{s.t.} \quad & \sum_{j=1}^{N} x_{ij} = B_i \ , \ \ x_{ij} \geq 0, \ \ j = 1, 2, \cdots, N \end{aligned} \tag{4}$$

Since $v_i(X)$ is strictly concave in $\boldsymbol{x}_i$, this maximization has a unique solution and can be solved using KKT conditions, for $j = 1, 2, \cdots, N$:

$$\begin{aligned} & -g_{ji} \cdot \mu_j'(\textstyle\sum_{k=1}^{N} x_{kj}) - \lambda_j + \nu = 0 \ , \\ & \lambda_j \cdot x_{ij} = 0, \quad x_{ij} \geq 0, \quad \textstyle\sum_{j=1}^{N} x_{ij} = B_i, \quad \lambda_j \geq 0 \ , \end{aligned} \tag{5}$$

where λ_j and ν are dual variables of the jth inequality and the equality constraints, respectively. We can simplify above KKT conditions and write the best response of agent i given action profile $\boldsymbol{x}_{-i}$ as follows:

$$x_{ij} = \left[(\mu_j')^{-1}(\frac{\nu}{g_{ji}}) - \sum_{k \in V - \{i\}} x_{kj} \right]^+ , \tag{6}$$

where $[a]^+ = \max\{0, a\}$. A pure-strategy Nash equilibrium (NE) of the public good provision game is a matrix of efforts $\overline{X} = [\overline{\boldsymbol{x}}_i, \overline{\boldsymbol{x}}_{-i}]^T$, for which

$$v_i(\overline{\boldsymbol{x}}_i, \overline{\boldsymbol{x}}_{-i}) \geq v_i(\boldsymbol{x}_i, \overline{\boldsymbol{x}}_{-i}), \forall \boldsymbol{x}_i \in S_i, \forall i, \tag{7}$$

where $S_i = \{(s_1, s_2, \cdots, s_N)|s_1 + s_2 + \cdots + s_N = B_i, s_j \geq 0 \ \forall j \in V\}$. We have the following result on the existence of NE in the game G_P.

Theorem 1 *The game G_P (public good provision with resource pooling) has at least one NE.*

Proof The strategy spaces S_i are nonempty, compact, and convex. Moreover, payoff functions $v_i(.)$ are continuous and concave in $\boldsymbol{x}_i$ for all $i \in V$. Using [8], there exists at least one pure-strategy Nash equilibrium. ■

Let v_i^* be the utility of agent i at the NE in game G_P. Then, in general, we may or may not have $v_i^* \geq v_i^o$; i.e., agents do not necessarily gain from resource pooling. In Sect. 5, we give some examples. There are however special cases where all agents obtain higher utility in G_P as we show below. We begin by characterizing the NE.

Lemma 1 *Let $\overline{X}$ be the action profile of agents in the NE of G_P. Then, $\overline{x}_{ij} \cdot \overline{x}_{ji} = 0,\ \forall i \neq j$.*

Proof Let us assume that $\overline{x}_{ij} \neq 0$. Note that $\mu_i'(\sum_{k=1}^N \overline{x}_{ki}) \leq g_{ji}\mu_j'(\sum_{k=1}^N \overline{x}_{kj})$ since otherwise agent i can improve its utility by increasing $\overline{x}_{ii}$ and decreasing $\overline{x}_{ij}$. Therefore, we have

$$\mu_i'(\textstyle\sum_{k=1}^N \overline{x}_{ki}) \leq g_{ji}\mu_j'(\sum_{k=1}^N \overline{x}_{kj}) \quad \rightarrow \quad g_{ij}\mu_i'(\sum_{k=1}^N \overline{x}_{ki}) < \mu_j'(\sum_{k=1}^N \overline{x}_{kj})\,, \tag{8}$$

which implies that $\overline{x}_{ji} = 0$ since otherwise agent j can improve its utility by decreasing $\overline{x}_{ji}$ and increasing $\overline{x}_{jj}$. As a result, $\overline{x}_{ij} \cdot \overline{x}_{ji} = 0$. ■

Lemma 1 says that if agent i improves agent j's utility by offering nonzero x_{ij}, then agent j will necessarily set $x_{ji} = 0$. Next, we provide some examples where the agents' utility improves in the NE of game G_P as compared to that without resource pooling (v_i^o).

Theorem 2 *1. Let $N = 2$. Then, $v_i^* \geq v_i^o,\ i = 1, 2$. In other words, in a network consisting of two agents, both achieve equal or higher utility at the NE under game G_P.*

2. Consider a network consisting of one parent (agent 1) and $N-1$ children: $g_{ij} = 0$ if $i \neq j$ and $i, j > 1$. Moreover, assume that $\mu_i(x) = \mu_j(x)$ and $B_i = B_j$ and $g_{i1} = g_{j1}$ and $g_{1i} = g_{1j}$ for all $i, j > 1$. Let v_i^* be the utility of the agent i at the symmetric Nash equilibrium where $\overline{x}_{i1} = \overline{x}_{j1}$ and $\overline{x}_{1i} = \overline{x}_{1j}$ for all $i, j > 1$. Then $v_i^* \geq v_i^o,\ \forall i \in V$.

Proof 1. Let $\overline{X}$ be the action profile of the agents in the NE. Then, by Lemma 1, we know that $\overline{x}_{12} = 0$ or $\overline{x}_{21} = 0$. Without loss of generality, let us assume that $\overline{x}_{12} = 0$. Therefore, $\overline{x}_{11} = B_1$. By the definition of NE, we have

$$v_2^* = \mu_2(\overline{x}_{22}) + g_{12}\mu_1(B_1 + \overline{x}_{21}) \geq \mu_2(B_2) + g_{12}\mu_1(B_1) = v_2^o\,. \tag{9}$$

Next, we show that $v_1^* \geq v_1^o$. We proceed by contradiction. Let us assume $\mu_1(B_1 + \overline{x}_{21}) + g_{21}\mu_2(\overline{x}_{22}) = v_1^* < v_1^o = \mu_1(B_1) + g_{21}\mu_2(B_2)$. This implies that decrease in $\overline{x}_{21}$ and increase in $\overline{x}_{22}$ improve the utility of agent 1 or equivalently $\mu_1'(B_1 + \overline{x}_{21}) < g_{21}\mu_2'(\overline{x}_{22})$. This is a contradiction since agent 1 can improve its utility by decreasing $\overline{x}_{11}$ and increasing $\overline{x}_{12}$. Therefore, $v_1^* \geq v_1^o$.

2. Consider a symmetric equilibrium where $\overline{x}_{i1} = \overline{x}_{j1}$ and $\overline{x}_{1i} = \overline{x}_{1j}$ for all $i, j > 1$. By Lemma 1, we know that $\overline{x}_{i1} = \overline{x}_{j1} = 0$ or $\overline{x}_{1i} = \overline{x}_{1j} = 0$. Therefore, we consider two different cases:
   Case 1: $\overline{x}_{i1} = \overline{x}_{j1} = 0$. Let $\overline{x}_{1i} = \overline{x}_{1j} = \overline{x}$. We have

$$v_1^* = \mu_1(\overline{x}_{11}) + (N-1)g_{21}\mu_2(B_2 + \overline{x}) \underbrace{\geq}_{\text{Definition of NE}} \mu_1(B_1) + (N-1)g_{21}\mu_2(B_2) = v_1^o \tag{10}$$

   To show that $v_2^* \geq v_2^o$, we proceed by contradiction. Let us assume that $v_2^* < v_2^o$. Then, we have

$$\begin{aligned} &\mu_2(B_2 + \overline{x}) + g_{12}\mu_1(\overline{x}_{11}) < \mu_2(B_2) + g_{12}\mu_1(B_1) \rightarrow \\ &\mu_2'(B_2 + \overline{x}) \leq g_{12}\mu_1'(\overline{x}_{11}) \rightarrow g_{21}\mu_2'(B_2 + \overline{x}) < \mu_1'(\overline{x}_{11}) \end{aligned} \tag{11}$$

   The last equation implies that agent 1 can improve its utility in NE by decreasing $\overline{x}$ and increasing $\overline{x}_{11}$. This is a contradiction and $v_2^* \geq v_2^o$.
   Case 2: $\overline{x}_{1i} = \overline{x}_{1j} = 0$. The proof is similar to case 1.

■

Theorem 2 provides two examples where game G_P can improve social welfare with resource pooling. While this cannot be guaranteed in general, we next design a mechanism guaranteed to induce a public good provision game with resource pooling where agents exert the socially optimal efforts at its NE.

4 A Mechanism with Socially Optimal Outcome

In this section, we present a taxation mechanism that implements the socially optimal solution at the NE of the game it induces in a decentralized setting. We begin by defining socially optimal strategies.

A socially optimal strategy is a strategy profile $X^* = \left[x_{ij}^*\right]_{N\times N}$ that solves the following optimization problem (total utility):

$$X^* \in \arg \max_{X=[\boldsymbol{x}_i;\boldsymbol{x}_{-i}]^T, \boldsymbol{x}_i \in S_i} \sum_{i=1}^{N} v_i(X) \,. \tag{12}$$

Notice that $v_i(X)$ is concave in X, but it is not necessarily strictly concave. Therefore, socially optimal effort profile X^* is not necessarily unique.

Generally, agents' actions at the NE of a game are not socially optimal; this is the case in the game G_P as we showed in the previous section. To induce socially optimal behavior, the approach of mechanism design is often used, whereby incentives are provided to induce a different game whose equilibrium coincides with the socially optimal solution.

We next describe such a mechanism based on taxation. In designing such a mechanism, we will assume that $\mu_i(.)$ and B_i are the private information of agent i unknown to the mechanism designer. Let t_i be the tax (punishment) levied on agent i by the mechanism designer; $t_i < 0$ is also referred to as a subsidy (reward). Agent i's utility after t_i is given by:

$$r_i(X, t_i) = v_i(X) - t_i \ . \tag{13}$$

A taxation mechanism is *budget-balanced* if the taxes at equilibrium are such that $\sum_{i=1}^{N} t_i = 0$; i.e., the mechanism designer neither seeks to make money from the agents nor injects money into the system. Under a budget-balanced mechanism, we have $\sum_{i=1}^{N} v_i(X) = \sum_{i=1}^{N} r_i(X, t_i)$. Our goal is to design a taxation mechanism which satisfies the following conditions:

1. The game induced by the mechanism has a NE.
2. The mechanism is socially optimal; i.e., it implements the socially optimal efforts at all NEs of the game it induces.
3. The mechanism is budget-balanced.
4. The mechanism is individual rational; i.e., the utility of agent i at the NE of the induced game is at least v_i^o for all i.[1]

Our mechanism is inspired by [17] and satisfies all of the above conditions.

A decentralized mechanism consists of a game form $(\mathcal{M}, h)$ where $\mathcal{M} := \prod_{i=1}^{N} \mathcal{M}_i$ and $\mathcal{M}_i$ is the set of all possible messages of agent/player i. Moreover, $h : \mathcal{M} \to \mathcal{A}$ is the outcome function and determines the effort profile and tax profile. Note that $\mathcal{A}$ is the space of all effort and tax profiles. The game form $(\mathcal{M}, h)$ together with utility functions $r_i(.)$ defines a game given by $(\mathcal{M}, h, \{r_i(.)\})$. We refer to this game as the game induced by the mechanism. A message profile of the decentralized mechanism $\overline{\boldsymbol{m}} = [\overline{m}_1, \overline{m}_2, \cdots, \overline{m}_N]$ is a Nash equilibrium of this game if

$$r_i(h(\overline{m}_i, \overline{\boldsymbol{m}}_{-i})) \geq r_i(h(m_i, \overline{\boldsymbol{m}}_{-i})), \ \forall m_i, \ \forall i \ . \tag{14}$$

The components of our decentralized mechanism are as follows.

The Message Space: Each agent i reports message $m_i = (\boldsymbol{x}^{(i)}, \boldsymbol{\pi}^{(i)})$, where $\boldsymbol{x}^{(i)}$ is a vector with $N \cdot (N-1)$ elements and is agent i's suggestion of all agents' effort profiles. In other words,

$$\boldsymbol{x}^{(i)} = \left[x_{12}^{(i)}, x_{13}^{(i)}, \cdots, x_{1N}^{(i)}, x_{21}^{(i)}, x_{23}^{(i)}, \cdots, x_{N(N-1)}^{(i)}\right], \ x_{jk}^{(i)} \in R, \ j \neq k, \tag{15}$$

[1]This is a weaker condition than voluntary participation, which requires that an agent's utility in the mechanism with everyone else is no less than his utility when unilaterally opting out. It has been shown in [13] that it is generally impossible to simultaneously achieve social optimality, weak budget balance, and voluntary participation.

where $x_{jk}^{(i)}$ is agent i's suggestion of agent j's effort to improve agent k's utility. Note that $\boldsymbol{x}^{(i)}$ has only $N \cdot (N-1)$ elements because $x_{jj}^{(i)}, j = 1, 2, \cdots, N$ are not in $\boldsymbol{x}^{(i)}$ but are completely determined by $\boldsymbol{x}^{(i)}$.

$\boldsymbol{\pi}^{(i)}$ is a price vector of real positive numbers used by the designer to determine the tax of each agent. Similar to $\boldsymbol{x}^{(i)}$, $\boldsymbol{\pi}^{(i)}$ has $N \cdot (N-1)$ elements:

$$\boldsymbol{\pi}^{(i)} = \left[\pi_{12}^{(i)}, \pi_{13}^{(i)}, \cdots, \pi_{1N}^{(i)}, \pi_{21}^{(i)}, \pi_{23}^{(i)}, \cdots, \pi_{N(N-1)}^{(i)}\right], \ \pi_{jk}^{(i)} \in R_+, \ j \neq k \,. \quad (16)$$

The Outcome Function: The outcome function determines the tax profile for each agent as well as investment profile $\hat{\boldsymbol{x}}(\boldsymbol{m})$. The investment profile $\hat{\boldsymbol{x}}(\boldsymbol{m})$ is calculated as follows,

$$\hat{\boldsymbol{x}}(\boldsymbol{m}) = \tfrac{1}{N} \textstyle\sum_{k=1}^{N} \boldsymbol{x}^{(k)}. \quad (17)$$

The amount of tax $\hat{t}_i$ agent i pays is given by

$$\begin{aligned} \hat{t}_i(\boldsymbol{m}) = {} & (\boldsymbol{\pi}^{(i+1)} - \boldsymbol{\pi}^{(i+2)})^T \hat{\boldsymbol{x}}(\boldsymbol{m}) \\ & + (\boldsymbol{x}^{(i)} - \boldsymbol{x}^{(i+1)})^T diag(\boldsymbol{\pi}^{(i)})(\boldsymbol{x}^{(i)} - \boldsymbol{x}^{(i+1)}) \\ & - (\boldsymbol{x}^{(i+1)} - \boldsymbol{x}^{(i+2)})^T diag(\boldsymbol{\pi}^{(i+1)})(\boldsymbol{x}^{(i+1)} - \boldsymbol{x}^{(i+2)}). \end{aligned} \quad (18)$$

Note that $N+1$ and $N+2$ are treated as 1 and 2 in (18). For the notational convenience and future use, we define $\hat{x}_{ii}(\boldsymbol{m}) = B_i - \sum_{k \in V - \{i\}} \hat{x}_{ik}, \forall i \in V$.

Note that Eq. (18) implies that the proposed mechanism is always budget-balanced because $\sum_{i=1}^{N} \hat{t}_i(\boldsymbol{m}) = 0, \forall \boldsymbol{m}$. Moreover, we have the following lemma on the tax term of the proposed mechanism at the NE of the induced game.

Lemma 2 *Let $\overline{\boldsymbol{m}}$ be a Nash equilibrium of the proposed mechanism, and $\overline{m}_i = (\overline{\boldsymbol{x}}^{(i)}, \overline{\boldsymbol{\pi}}^{(i)})$ and $\underline{\boldsymbol{x}} = \hat{\boldsymbol{x}}(\overline{\boldsymbol{m}})$. Then,*

$$\hat{t}_i(\overline{\boldsymbol{m}}) = (\overline{\boldsymbol{\pi}}^{(i+1)} - \overline{\boldsymbol{\pi}}^{(i+2)})^T \cdot \underline{\boldsymbol{x}} \ \forall i \quad (19)$$

Proof The proof is provided in appendix. ■

Lemma 2 implies that both terms $(\boldsymbol{x}^{(i)} - \boldsymbol{x}^{(i+1)})^T diag(\boldsymbol{\pi}^{(i)})(\boldsymbol{x}^{(i)} - \boldsymbol{x}^{(i+1)})$ and $(\boldsymbol{x}^{(i+1)} - \boldsymbol{x}^{(i+2)})^T diag(\boldsymbol{\pi}^{(i+1)})(\boldsymbol{x}^{(i+1)} - \boldsymbol{x}^{(i+2)})$ in (18) vanish at the equilibrium of the proposed mechanism. The inclusion of these two terms is necessary to make sure that the mechanism implements the socially optimal effort profile at each NE.

We next show that the proposed mechanism is individually rational.

Theorem 3 *Let $\overline{\boldsymbol{m}}$ be a NE of the proposed mechanism. Then, the agents achieve higher utility at the NE than their outside option if they all opt out. That is, $r_i(\hat{X}(\overline{\boldsymbol{m}}), \hat{t}_i(\overline{\boldsymbol{m}})) \geq v_i^o, \ \forall i \in V$, where $\hat{X}(\overline{\boldsymbol{m}}) = [\hat{x}_{ij}]_{N \times N}$ is the matrix of agents' effort.*

Proof Proof is provided in appendix. ■

We are now ready to prove the main theorem about socially optimal mechanism with resource pooling. The next theorem shows that if $\hat{X}(\overline{\boldsymbol{m}})$ is the outcome of the proposed mechanism at a Nash equilibrium, then $\hat{X}(\overline{\boldsymbol{m}})$ is a solution to the optimization problem (12). In other words, the NEs of the game induced by proposed mechanism implement a socially optimal effort profile. Note that the game does not necessarily have a unique NE in terms of the messages, and the outcome is one of the socially optimal solutions to optimization problem (12).

Theorem 4 *Let $\overline{\boldsymbol{m}}$ be a NE of game $(\mathcal{M}, h, \{r_i(.)\})$ induced by the proposed mechanism. Then, $\hat{X}(\overline{\boldsymbol{m}})$ is a optimal solution to optimization problem* (12)*.*

Proof Let $\overline{\boldsymbol{m}} = [(\overline{\boldsymbol{x}}^{(1)}, \overline{\boldsymbol{\pi}}^{(1)}), \cdots, (\overline{\boldsymbol{x}}^{(N)}, \overline{\boldsymbol{\pi}}^{(N)})]$ be a NE of the proposed mechanism. By definition of the Nash equilibrium, we can write,

$$r_i(\hat{X}((\boldsymbol{x}^{(i)}, \boldsymbol{0}), \overline{\boldsymbol{m}}_{-i}), t_i((\boldsymbol{x}^{(i)}, \boldsymbol{0}), \overline{\boldsymbol{m}}_{-i})) \leq r_i(\hat{X}(\overline{\boldsymbol{m}}), \hat{t}_i(\overline{\boldsymbol{m}})) \tag{20}$$

By Lemma 2 and Eq. (18), $t_i((\boldsymbol{x}^{(i)}, \boldsymbol{0}), \overline{\boldsymbol{m}}_{-i}) = (\overline{\boldsymbol{\pi}}^{(i+1)} - \overline{\boldsymbol{\pi}}^{(i+2)})^T \hat{\boldsymbol{x}}((\boldsymbol{x}^{(i)}, \boldsymbol{0}), \overline{\boldsymbol{m}}_{-i}))$. Therefore, (20) can be written as follows,

$$\begin{aligned} &v_i(A(\hat{\boldsymbol{x}}((\boldsymbol{x}^{(i)}, \boldsymbol{0}), \overline{\boldsymbol{m}}_{-i}))) - (\overline{\boldsymbol{\pi}}^{(i+1)} - \overline{\boldsymbol{\pi}}^{(i+2)})^T \hat{\boldsymbol{x}}((\boldsymbol{x}^{(i)}, \boldsymbol{0}), \overline{\boldsymbol{m}}_{-i})) \leq \\ &v_i(A(\hat{\boldsymbol{x}}(\overline{\boldsymbol{m}}))) - (\overline{\boldsymbol{\pi}}^{(i+1)} - \overline{\boldsymbol{\pi}}^{(i+2)})^T \underline{\boldsymbol{x}} \ \forall \boldsymbol{x}^{(i)} \end{aligned} \tag{21}$$

where

$$A(\boldsymbol{x}) = \begin{bmatrix} \left[B_1 - \sum_{k \in V-\{1\}} x_{1k}\right] & x_{12} & \cdots & x_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \cdots & \left[B_N - \sum_{k \in V-\{N\}} x_{Nk}\right] \end{bmatrix} \tag{22}$$

Substituting $\boldsymbol{x} = \frac{1}{N}(\boldsymbol{x}^{(i)} + \sum_{k \in V-\{i\}} \overline{\boldsymbol{x}}^{(k)})$ and using (21), we have

$$\underline{\boldsymbol{x}} = \arg\max_{\{\boldsymbol{x} \in R^{N(N-1)}, A(\boldsymbol{x}) \in S\}} v_i(A(\boldsymbol{x})) - (\overline{\boldsymbol{\pi}}^{(i+1)} - \overline{\boldsymbol{\pi}}^{(i+2)})^T \boldsymbol{x} \ , \tag{23}$$

where S is the feasible set of effort profiles: $S = \{X = \left[x_{ij}\right] \in R^{N \times N} | \sum_{j=1}^{N} x_{ij} = B_i, x_{ij} \geq 0, \ \forall i, j \in V\}$. Because the optimization in (23) is convex, KKT conditions are necessary and sufficient for the optimality of $\underline{\boldsymbol{x}}$. The KKT conditions for (23) are given by:

$$\begin{aligned} &(\overline{\boldsymbol{\pi}}^{(i+1)} - \overline{\boldsymbol{\pi}}^{(i+2)}) - \nabla_{\boldsymbol{x}} v_i(A(\underline{\boldsymbol{x}})) - \boldsymbol{\lambda}^i + \boldsymbol{\theta}^i = 0, \\ &(\boldsymbol{\lambda}^i)^T \underline{\boldsymbol{x}} = 0, \\ &\theta^i_j(-B_j + \textstyle\sum_{k \in V-\{j\}} \underline{x}_{jk}) = 0, \\ &\boldsymbol{\lambda}^i \geq 0, \\ &\boldsymbol{\theta}^i \geq 0 \ , \end{aligned} \tag{24}$$

where $\boldsymbol{\lambda}^i = \left[\lambda^i_{12}, \lambda^i_{13}, \cdots, \lambda^i_{1N}, \lambda^i_{21}, \lambda^i_{23}, \cdots, \lambda^i_{(N-1)N}\right]^T \in R_+^{N^2-N}$ and $\boldsymbol{\theta}^i = \left[\underbrace{\theta^i_1, \cdots, \theta^i_1}_{N-1\ times}, \underbrace{\theta^i_2, \cdots, \theta^i_2}_{N-1\ times}, \cdots, \underbrace{\theta^i_N, \cdots, \theta^i_N}_{N-1\ times}\right]^T \in R_+^{N^2-N}$. Here, λ^i_{jk} is the dual variable corresponding to constraint $\underline{x}_{ij} \geq 0$ and θ^i_j is the dual variable corresponding to constraint $\sum_{k \in V-\{j\}} \underline{x}_{jk} \leq B_j$.

Summing (24) over all i in V we get

$$\begin{aligned} & -\left(\textstyle\sum_{i\in V} \nabla_{\boldsymbol{x}} v_i(A(\underline{\boldsymbol{x}}))\right) - \boldsymbol{\lambda} + \boldsymbol{\theta} = 0 \\ & \boldsymbol{\lambda}^T \underline{\boldsymbol{x}} = 0, \\ & \theta_j(-B_j + \textstyle\sum_{k\in V-\{j\}}, \overline{x}_{jk}) = 0, \\ & \boldsymbol{\lambda} \geq 0, \\ & \boldsymbol{\theta} \geq 0\,. \end{aligned} \tag{25}$$

where $\boldsymbol{\theta} = \sum_{i\in V} \boldsymbol{\theta}^i = \left[\underbrace{\theta_1, \cdots, \theta_1}_{N-1\ times}, \underbrace{\theta_2, \cdots, \theta_2}_{N-1\ times}, \cdots, \underbrace{\theta_N, \cdots, \theta_N}_{N-1\ times}\right]^T, \theta_j = \sum_{i\in V} \theta^i_j$ and $\boldsymbol{\lambda} = \sum_{i\in V} \boldsymbol{\lambda}^i$. Note that (25) is the KKT conditions for following convex optimization problem:

$$\begin{aligned} & \max_{\{\boldsymbol{x}\in R^{N(N-1)}\}} \textstyle\sum_{i\in V} v_i(A(\boldsymbol{x})) \\ & s.t., \\ & x_{ij} \geq 0\ \forall i \neq j \\ & \textstyle\sum_{k\in V-\{j\}} x_{jk} \leq B_j\ \forall j \in V\,. \end{aligned} \tag{26}$$

Because (26) is a convex problem, then KKT conditions (25) are necessary and sufficient for the optimal solution of (26). As a result, $\underline{\boldsymbol{x}}$ is a socially optimal effort profile. ■

As we can see from the proof of Theorem 4, socially optimal effort profile $\hat{X}(\overline{\boldsymbol{m}})$ is individually optimal at the NE of the induced game. That is,

$$\begin{aligned} & \hat{X}(\overline{\boldsymbol{m}}) = A(\underline{x}), \text{ where,} \\ & \underline{\boldsymbol{x}} = \arg\max_{\{\boldsymbol{x}\in R^{N(N-1)}, A(\boldsymbol{x})\in S\}} v_i(A(\boldsymbol{x})) - (\overline{\boldsymbol{\pi}}^{(i+1)} - \overline{\boldsymbol{\pi}}^{(i+2)})^T \boldsymbol{x}\,. \end{aligned} \tag{27}$$

Theorem 4 implies that the NEs of the game induced by the proposed mechanism implement a socially optimal effort profile. We next show the converse of Theorem 4, i.e., given any socially optimal effort profile X^*, the induced game has at least one NE which implements effort profile X^*.

Theorem 5 *Let X^* be a socially optimal effort profile. Then, there is a Nash equilibrium of game $(\mathcal{M}, h(.), \{r_i(.)\})$ induced by the proposed mechanism which implements socially optimal effort profile X^*.*

Proof The proof is provided in appendix. ■

Theorem 4 and 5 together imply that the game induced by the proposed mechanism always has at least a Nash equilibrium and each Nash equilibrium implements a socially optimal effort profile.

5 Numerical Example

5.1 An Example of Three Interdependent Agents

In this section, we provide an example of three interdependent agents. Consider the following parameters.

$$N = 3, g_{ij} = e^{-1},\ \forall i \neq j,\ \mu_1(y) = \mu_2(y) = \mu_3(y) = -e^{-y}$$

$$B_1 = 5, B_2 = B_3 = 1\ .$$

The utility of the agents without resource pooling is given by

$$\begin{aligned} v_1^o &= -e^{-5} - 2e^{-2} \\ v_2^o &= -e^{-1} - e^{-6} - e^{-2} \\ v_3^o &= -e^{-1} - e^{-6} - e^{-2}\ . \end{aligned} \tag{28}$$

If we consider G_P, then the best response of agent i is given by

$$Br_1(\boldsymbol{x}_{-1}) = \begin{bmatrix} [-\ln \nu_1 - x_{21} - x_{31}]^+ \\ [-\ln \nu_1 - 1 - x_{22} - x_{32}]^+ \\ [-\ln \nu_1 - 1 - x_{33} - x_{23}]^+ \end{bmatrix} = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}, \tag{29}$$

where ν_1 is a nonnegative number and is determined by the budget constraint $x_{11} + x_{12} + x_{13} = B_1$. Similarly, we can write the best-response function of the other agents as follows.

$$Br_2(\boldsymbol{x}_{-2}) = \begin{bmatrix} [-\ln \nu_2 - 1 - x_{11} - x_{31}]^+ \\ [-\ln \nu_2 - x_{12} - x_{32}]^+ \\ [-\ln \nu_2 - 1 - x_{33} - x_{13}]^+ \end{bmatrix} = \begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}, \tag{30}$$

$$Br_3(\boldsymbol{x}_{-3}) = \begin{bmatrix} [-\ln \nu_3 - 1 - x_{21} - x_{11}]^+ \\ [-\ln \nu_3 - 1 - x_{22} - x_{12}]^+ \\ [-\ln \nu_3 - x_{13} - x_{23}]^+ \end{bmatrix} = \begin{bmatrix} x_{31} \\ x_{32} \\ x_{33} \end{bmatrix}. \tag{31}$$

The fixed point of these three best-response mappings is given by

$$\begin{bmatrix}\overline{x}_{11}\\ \overline{x}_{12}\\ \overline{x}_{13}\end{bmatrix}=\begin{bmatrix}3\\1\\1\end{bmatrix},\quad \begin{bmatrix}\overline{x}_{21}\\ \overline{x}_{22}\\ \overline{x}_{23}\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad \begin{bmatrix}\overline{x}_{31}\\ \overline{x}_{32}\\ \overline{x}_{33}\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}, \tag{32}$$

and the agents' utility at the NE of game G_P is given by

$$\begin{aligned}&v_1(\overline{X}) = -3e^{-3}\\ &v_2(\overline{X}) = -e^{-2} - e^{-4} - e^{-3}\\ &v_3(\overline{X}) = -e^{-2} - e^{-4} - e^{-3}\\ &\overline{X} = \left[\overline{x}_{i,j}\right]_{3\times 3} .\end{aligned} \tag{33}$$

It is easy to check that $v_i(\overline{X}) \geq v_i^o$. Therefore, in this example the utility of agents at the NE of the public good game with resource pooling is higher than that without resource pooling.

It is also easy to calculate the socially optimal efforts:

$$\begin{aligned}&X^* \in \arg\max_{X\in S} v_1(X) + v_2(X) + v_3(X) =\\ &-(1+2e^{-1})\cdot(\exp\{-x_{11}-x_{21}-x_{31}\}+\\ &\exp\{-x_{12}-x_{22}-x_{32}\} + \exp\{-x_{13}-x_{23}-x_{33}\})\\ &\boldsymbol{x}_1^* = \begin{bmatrix}\frac{5}{3}\\ \frac{5}{3}\\ \frac{5}{3}\end{bmatrix},\quad \boldsymbol{x}_2^* = \begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\\ \frac{1}{3}\end{bmatrix},\quad \boldsymbol{x}_3^* = \begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\\ \frac{1}{3}\end{bmatrix}.\end{aligned} \tag{34}$$

This is one of the socially optimal strategies, and there exits a NE of the induced game which implements this socially optimal effort profile.

5.2 An Example where Resource Pooling Worsens Social Welfare

Consider an example of $N = 12$ interdependent agent, where $g_{21} = e^{-1}$ and $g_{1j} = e^{-1}, \forall j > 2, g_{ii} = 1, \forall i$, and all other edge weights are zero. Moreover, consider the following parameters:

$$\mu_i(y) = -e^{-y},\quad \forall i$$
$$B_1 = 3, B_i = 0,\quad \forall i \geq 2 .$$

The utility of the agents when the first agent does not pool his resource is:

$$\begin{aligned}&v_1^o = -e^{-3} - e^{-1}\\ &v_2^o = -1\\ &v_i^o = -1 - e^{-4},\ \forall i \geq 3 .\end{aligned} \tag{35}$$

At the same time, it is easy to see that if the first agent chooses $\overline{x}_{11} = 2$ and $\overline{x}_{12} = 1$, then his utility is maximized. We have the following utilities at the NE after this resource pooling:

$$\begin{aligned} v_1(\overline{X}) &= -2e^{-2} \\ v_2(\overline{X}) &= -e^{-1} \\ v_i(\overline{X}) &= -e^{-3} - 1,\ \forall i \geq 3 \\ \overline{x}_{11} &= 2,\ \overline{x}_{12} = 1,\ \overline{x}_{ij} = 0,\ \forall i, j \text{ and } (i, j) \neq (1, 1) \text{ or } (1, 2)\ . \end{aligned} \tag{36}$$

In this example, $\sum_{i=1}^{12} v_i(\overline{X}) < \sum_{i=1}^{12} v_i^o$. That is, resource pooling worsens the social welfare without the proposed mechanism. Incidentally, $x_{11}^* = 3$ and $x_{ij}^* = 0,\ \forall (i, j) \in V \times V - \{(1, 1)\}$ constitute the socially optimal effort profile.

6 Conclusion

We studied a public good provision game with resource pooling. We showed that resource pooling does not necessarily improve social welfare when agents act selfishly. We then presented a tax-based mechanism which incentivizes agents to pool their resources in a desirably manner. This mechanism is budget-balanced and implements the socially optimal solution at the Nash equilibrium of the game it induces.

Appendix

Proof (*Lemma* 2) Let $\overline{\boldsymbol{m}}$ be a Nash equilibrium of the game induced by proposed mechanism, and $\overline{m}_i = (\overline{\boldsymbol{x}}^{(i)}, \overline{\boldsymbol{\pi}}^{(i)})$. We need to show that the following term is equal to zero at NE:

$$\begin{aligned} &(\overline{\boldsymbol{x}}^{(i)} - \overline{\boldsymbol{x}}^{(i+1)})^T diag(\overline{\boldsymbol{\pi}}^{(i)})(\overline{\boldsymbol{x}}^{(i)} - \overline{\boldsymbol{x}}^{(i+1)}) \\ &-(\overline{\boldsymbol{x}}^{(i+1)} - \overline{\boldsymbol{x}}^{(i+2)})^T diag(\overline{\boldsymbol{\pi}}^{(i+1)})(\overline{\boldsymbol{x}}^{(i+1)} - \overline{\boldsymbol{x}}^{(i+2)})\ . \end{aligned} \tag{37}$$

Because $\overline{\boldsymbol{m}}$ is the Nash equilibrium, we have

$$r_i(\hat{X}(m_i, \overline{\boldsymbol{m}}_{-i}), \hat{t}_i(m_i, \overline{\boldsymbol{m}}_{-i})) \leq r_i(\hat{X}(\overline{\boldsymbol{m}}), \hat{t}_i(\overline{\boldsymbol{m}})), \forall m_i \in \mathcal{M}_i\ . \tag{38}$$

We substitute $m_i = (\overline{\boldsymbol{x}}^{(i)}, \boldsymbol{\pi}^{(i)})$ in (38). Using (17) and (18), we have,

$$\begin{aligned} &r_i(\hat{X}((\overline{\boldsymbol{x}}^{(i)}, \boldsymbol{\pi}^{(i)}), \overline{\boldsymbol{m}}_{-i}), \hat{t}_i((\overline{\boldsymbol{x}}^{(i)}, \boldsymbol{\pi}^{(i)}), \overline{\boldsymbol{m}}_{-i})) = \\ &r_i(\hat{X}(\overline{\boldsymbol{m}}), \hat{t}_i((\overline{\boldsymbol{x}}^{(i)}, \boldsymbol{\pi}^{(i)}), \overline{\boldsymbol{m}}_{-i})) \leq r_i(\hat{X}(\overline{\boldsymbol{m}}), \hat{t}_i(\overline{\boldsymbol{m}})), \forall \boldsymbol{\pi}^{(i)} \in R_+^{N(N-1)}\ . \end{aligned} \tag{39}$$

Because $r_i(.,.)$ is a decreasing function in t_i, (39) implies that,

$$\hat{t}_i((\overline{\boldsymbol{x}}^{(i)}, \boldsymbol{\pi}^{(i)}), \overline{\boldsymbol{m}}_{-i}) \geq \hat{t}_i(\overline{\boldsymbol{m}}), \forall \boldsymbol{\pi}^{(i)} \in R_+^{N(N-1)} . \tag{40}$$

In other words,

$$\begin{aligned}
&(\overline{\boldsymbol{\pi}}^{(i+1)} - \overline{\boldsymbol{\pi}}^{(i+2)})^T \underline{\boldsymbol{x}} \\
&+(\overline{\boldsymbol{x}}^{(i)} - \overline{\boldsymbol{x}}^{(i+1)})^T diag(\overline{\boldsymbol{\pi}}^{(i)})(\overline{\boldsymbol{x}}^{(i)} - \overline{\boldsymbol{x}}^{(i+1)}) \\
&-(\overline{\boldsymbol{x}}^{(i+1)} - \overline{\boldsymbol{x}}^{(i+2)})^T diag(\overline{\boldsymbol{\pi}}^{(i+1)})(\overline{\boldsymbol{x}}^{(i+1)} - \overline{\boldsymbol{x}}^{(i+2)}) \leq \\
&(\overline{\boldsymbol{\pi}}^{(i+1)} - \overline{\boldsymbol{\pi}}^{(i+2)})^T \underline{\boldsymbol{x}} \\
&+(\overline{\boldsymbol{x}}^{(i)} - \overline{\boldsymbol{x}}^{(i+1)})^T diag(\boldsymbol{\pi}^{(i)})(\overline{\boldsymbol{x}}^{(i)} - \overline{\boldsymbol{x}}^{(i+1)}) \\
&-(\overline{\boldsymbol{x}}^{(i+1)} - \overline{\boldsymbol{x}}^{(i+2)})^T diag(\overline{\boldsymbol{\pi}}^{(i+1)})(\overline{\boldsymbol{x}}^{(i+1)} - \overline{\boldsymbol{x}}^{(i+2)}) , \\
&\forall \boldsymbol{\pi}^{(i)} \in R_+^{N\cdot(N-1)} .
\end{aligned} \tag{41}$$

By simplifying the above equation, we have

$$(\overline{\boldsymbol{x}}^{(i)} - \overline{\boldsymbol{x}}^{(i+1)})^T diag(\boldsymbol{\pi}^{(i)} - \overline{\boldsymbol{\pi}}^{(i)})(\overline{\boldsymbol{x}}^{(i)} - \overline{\boldsymbol{x}}^{(i+1)}) \geq 0 , \forall \boldsymbol{\pi}^{(i)} \in R_+^{N\cdot(N-1)} . \tag{42}$$

Because the above equation is valid for all $\boldsymbol{\pi}^{(i)} \in R_+^{N(N-1)}$, it implies

$$(\overline{\boldsymbol{x}}^{(i)} - \overline{\boldsymbol{x}}^{(i+1)})^T diag(\overline{\boldsymbol{\pi}}^{(i)})(\overline{\boldsymbol{x}}^{(i)} - \overline{\boldsymbol{x}}^{(i+1)}) = 0, \ \forall i \in V . \tag{43}$$

Therefore, at the NE we have

$$\hat{t}_i(\overline{\boldsymbol{m}}) = (\overline{\boldsymbol{\pi}}^{(i+1)} - \overline{\boldsymbol{\pi}}^{(i+2)})^T \underline{\boldsymbol{x}}, \ \forall i \in V . \tag{44}$$

■

Proof (*Theorem* 3) Let $\overline{\boldsymbol{m}}$ be a NE of the proposed mechanism, and $\overline{m}_i = (\overline{\boldsymbol{x}}^{(i)}, \overline{\boldsymbol{\pi}}^{(i)})$ and $\underline{\boldsymbol{x}} = \hat{\boldsymbol{x}}(\overline{\boldsymbol{m}})$. By definition, we have

$$\begin{aligned}
&r_i(\hat{X}((\boldsymbol{x}^{(i)}, \boldsymbol{\pi}^{(i)}), \overline{\boldsymbol{m}}_{-i}), \hat{t}_i((\boldsymbol{x}^{(i)}, \boldsymbol{\pi}^{(i)}), \overline{\boldsymbol{m}}_{-i})) \leq r_i(\hat{X}(\overline{\boldsymbol{m}}), \hat{t}_i(\overline{\boldsymbol{m}})), \\
&\forall m_i = (\boldsymbol{x}^{(i)}, \boldsymbol{\pi}^{(i)}) \in \mathcal{M}_i
\end{aligned} \tag{45}$$

Let $\tilde{\boldsymbol{x}}^{(i)}$ be a vector such that $\frac{1}{N}(\tilde{\boldsymbol{x}}^{(i)} + \sum_{k\in V-\{i\}} \overline{\boldsymbol{x}}^{(k)}) = 0$. Moreover, we set $\boldsymbol{\pi}^{(i)} = \mathbf{0}$. By Lemma 2, we have

$$\begin{aligned}
&\hat{\boldsymbol{x}}((\tilde{\boldsymbol{x}}^{(i)}, \mathbf{0}), \overline{\boldsymbol{m}}_{-i}) = \mathbf{0} \\
&\hat{X}((\tilde{\boldsymbol{x}}^{(i)}, \mathbf{0}), \overline{\boldsymbol{m}}_{-i}) = diag(B_1, B_2, \cdots, B_N) \\
&\hat{t}_i((\tilde{\boldsymbol{x}}^{(i)}, \mathbf{0}), \overline{\boldsymbol{m}}_{-i}) = \mathbf{0} \\
&r_i(\hat{X}((\tilde{\boldsymbol{x}}^{(i)}, \mathbf{0}), \overline{\boldsymbol{m}}_{-i}), \hat{t}_i((\tilde{\boldsymbol{x}}^{(i)}, \mathbf{0}), \overline{\boldsymbol{m}}_{-i})) = v_i^o .
\end{aligned} \tag{46}$$

Equations (45) and (46) together imply that $r_i(\hat{X}(\overline{\boldsymbol{m}}), \hat{t}_i(\overline{\boldsymbol{m}})) \geq v_i^o$. ■

Proof (*Theorem* 5) Let us assume X^* is a socially optimal effort profile. Let $\underline{\boldsymbol{x}} = [x_{12}^*, x_{13}^*, \cdots, x_{1N}^*, x_{21}^*, x_{23}^* \cdots, x_{N(N-1)}^*]$. First we show that there is vector $\bar{\boldsymbol{l}}_i$ such that

$$\underline{\boldsymbol{x}} \in \arg \max_{\boldsymbol{x} \in R^{N(N-1)}, A(\boldsymbol{x}) \in S} -\bar{\boldsymbol{l}}_i^T \boldsymbol{x} + v_i(A(\boldsymbol{x})) \ . \tag{47}$$

As $A(\underline{\boldsymbol{x}})$ is the socially optimal effort profile, we have

$$\begin{aligned}
&\underline{\boldsymbol{x}} = \arg\max_{\{\boldsymbol{x} \in R^{N(N-1)}, A(\boldsymbol{x}) \in S\}} \textstyle\sum_{i=1}^N v_i(A(\boldsymbol{x})) \rightarrow \text{KKT Conditions:}\\
&-\left(\textstyle\sum_{i \in V} \nabla_{\boldsymbol{x}} v_i(A(\underline{\boldsymbol{x}}))\right) - \boldsymbol{\lambda} + \boldsymbol{\theta} = 0\\
&\boldsymbol{\lambda}^T \underline{\boldsymbol{x}} = 0,\\
&\theta_j(-B_j + \textstyle\sum_{k \in V - \{j\}} \underline{x}_{jk}) = 0\\
&\boldsymbol{\lambda} \geq 0\\
&\boldsymbol{\theta} = \Big[\underbrace{\theta_1, \cdots, \theta_1}_{N-1\ times}, \underbrace{\theta_2, \cdots, \theta_2}_{N-1\ times}, \cdots, \underbrace{\theta_N, \cdots, \theta_N}_{N-1\ times}\Big]^T \geq 0 \ .
\end{aligned} \tag{48}$$

We can define $\bar{\boldsymbol{l}}_i = \nabla_{\boldsymbol{x}} v_i(A(\underline{\boldsymbol{x}})) + \boldsymbol{\lambda}/N - \boldsymbol{\theta}/N$. Then we have

$$\bar{\boldsymbol{l}}_i - \nabla_{\boldsymbol{x}} v_i(A(\underline{\boldsymbol{x}})) - \boldsymbol{\lambda}/N + \boldsymbol{\theta}/N = \boldsymbol{0} \ , \tag{49}$$

which implies that $\underline{\boldsymbol{x}}, \boldsymbol{\lambda}/N, \boldsymbol{\theta}/N$ satisfies the KKT conditions for the following optimization problem:

$$\arg\max_{\{\boldsymbol{x} \in R^{N(N-1)}, A(\boldsymbol{x}) \in S\}} -\bar{\boldsymbol{l}}_i^T \boldsymbol{x} + v_i(A(\boldsymbol{x})) \ . \tag{50}$$

As the above optimization problem is convex and the KKT conditions are necessary and sufficient for optimality, $\underline{\boldsymbol{x}}$ is the solution to (50).

Now let us assume that we have already found $\bar{\boldsymbol{l}}_i, \forall i \in V$. Consider following system of equations,

$$\begin{aligned}
&\tfrac{1}{N} \textstyle\sum_{i=1}^N \overline{\boldsymbol{x}}^{(i)} = \underline{\boldsymbol{x}} && (51.a)\\
&\overline{\boldsymbol{\pi}}^{(i+1)} - \overline{\boldsymbol{\pi}}^{(i+2)} = \bar{\boldsymbol{l}}_i, \ i = 1, \cdots, N && (51.b)\\
&(\overline{\boldsymbol{x}}^{(i)} - \overline{\boldsymbol{x}}^{(i+1)})^T diag(\overline{\boldsymbol{\pi}}^{(i)})(\overline{\boldsymbol{x}}^{(i)} - \overline{\boldsymbol{x}}^{(i+1)}) = 0, \ i = 1, \cdots, N && (51.c)\\
&\overline{\boldsymbol{\pi}}^{(i)} \geq \boldsymbol{0}, \ i = 1, \cdots, N && (51.d)
\end{aligned} \tag{51}$$

First, we show that the above system of equations has at least one solution.

If we set $\overline{\boldsymbol{x}}^{(i)} = \underline{\boldsymbol{x}}$, then Eqs. (51.$a$), (51.$c$) are satisfied. Moreover, the summation of left-hand side and right-hand side of (51.b) is zero which implies that one of the equations of type (51.b) is redundant. Therefore, if we choose an arbitrary value for $\overline{\boldsymbol{\pi}}^{(1)}$, then $\overline{\boldsymbol{\pi}}^{(2)}, \overline{\boldsymbol{\pi}}^{(3)}, \cdots, \overline{\boldsymbol{\pi}}^{(N)}$ can be determined accordingly based on (51.b).

Moreover, notice that if we add all $\overline{\boldsymbol{\pi}}^{(i)}$ by a constant vector $\boldsymbol{c}$, then they still satisfy (51.a), (51.b), (51.c). Therefore, we can select an appropriate constant vector $\boldsymbol{c}$ to satisfy (51.d).

Now, we show the solution introduced above is a Nash equilibrium of the proposed mechanism. We chose $\bar{\boldsymbol{l}}_i$ such that it satisfies the following:

$$\overline{\boldsymbol{x}}^{(i)} = \underline{\boldsymbol{x}} \in \arg\max_{\boldsymbol{x} \in R^{N\cdot(N-1)}} -\bar{\boldsymbol{l}}_i^T \boldsymbol{x} + v_i(A(\boldsymbol{x})) \ . \tag{52}$$

We use the following change of variable for the above optimization problem: $N\boldsymbol{x} - \sum_{j\in V-\{i\}} \overline{\boldsymbol{x}}^{(j)} = \boldsymbol{x}^{(i)}$. We have

$$\begin{aligned}\overline{\boldsymbol{x}}^{(i)} \in \underline{\boldsymbol{x}} \in \arg\max_{\boldsymbol{x}^{(i)} \in R^{N\cdot(N-1)}} &-\bar{\boldsymbol{l}}_i^T \tfrac{1}{N}(\boldsymbol{x}^{(i)} + \textstyle\sum_{j\in V-\{i\}} \overline{\boldsymbol{x}}^{(j)}) \\ &+ v_i(A(\tfrac{1}{N}(\boldsymbol{x}^{(i)} + \textstyle\sum_{j\in V-\{i\}} \overline{\boldsymbol{x}}^{(j)}))) \ .\end{aligned} \tag{53}$$

By (51.c) and the fact that the users' utility function is decreasing in tax, we have

$$\begin{aligned}(\overline{\boldsymbol{x}}^{(i)}, \overline{\boldsymbol{\pi}}^{(i)}) \in \arg\max_{\{\boldsymbol{x}^{(i)} \in R^{N\cdot(N-1)}, \boldsymbol{\pi}^{(i)}\}} &-\bar{\boldsymbol{l}}_i^T \tfrac{1}{N}(\boldsymbol{x}^{(i)} + \textstyle\sum_{j\in V-\{i\}} \overline{\boldsymbol{x}}^{(j)}) \\ &(\overline{\boldsymbol{x}}^{(i+1)} - \overline{\boldsymbol{x}}^{(i+2)})^T diag(\overline{\boldsymbol{\pi}}^{(i+1)})(\overline{\boldsymbol{x}}^{(i+1)} - \overline{\boldsymbol{x}}^{(i+2)})^T \\ &-(\boldsymbol{x}^{(i)} - \overline{\boldsymbol{x}}^{(i+1)})^T diag(\boldsymbol{\pi}^{(i)})(\boldsymbol{x}^{(i)} - \overline{\boldsymbol{x}}^{(i+1)})^T \\ &+ v_i(A(\tfrac{1}{N}(\boldsymbol{x}^{(i)} + \textstyle\sum_{j\in V-\{i\}} \overline{\boldsymbol{x}}^{(j)})))\end{aligned} \tag{54}$$

The last equation implies that the solution to (51) is the fixed point of the best-response mapping. Therefore, the solution to (51) is a NE of the proposed mechanism. ■

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